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The spectral geometry of the canonical Riemannian submersion of a compact Lie group

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Abstract

Let G be a compact connected Lie group which is equipped with a bi-invariant Riemannian metric. Let $m(x, y) = xy$ be the multiplication operator. We show the associated fibration $m : G \times G \to G$ is a Riemannian submersion with totally geodesic fibers and we study the spectral geometry of this submersion. We show that the pull-backs of eigenforms on the base have finite Fourier series on the total space and we give examples where arbitrarily many Fourier coefficients can be non-zero. We give necessary and sufficient conditions for the pull-back of a form on the base to be harmonic on the total space.
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1. Introduction

The spectral geometry of Riemannian manifolds has been studied extensively; compact Lie groups play a central role in this investigation. For example, work of Schueth [13] shows that there are non-trivial isospectral families of left invariant metrics on compact Lie groups, although any such family which includes a bi-invariant metric is necessarily trivial; this work has been extended by Proctor [11]. Riemannian submersions of Lie groups with totally geodesic fibers have been studied by Ranjan [12]. We refer the reader to [3] for a further discussion of the spectral geometry of Riemannian submersions.

There are many instances in the physics literature where non-bijective canonical transformations (i.e. Riemannian submersions) have been investigated. Boiteux [1] studied the Coulomb potential in two and three dimensions and noted that "in quantum mechanics, those transformations connect operators with different spectra which as such cannot be deduced from one another by unitary transformations". Recent work by Kibler [7] discusses the

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Kustaanheimo–Steiffel transformation in classical mechanics for regularizing the Kepler problem; this is a non-bijective canonical transformation which is quadratic and based on the Hopf fibration. We also refer the reader to Lambert and Kibler [8] for related work and for a more extensive review of the classical literature in this subject than is possible here.

In this paper, we shall study the spectral geometry of the multiplication map $m : G \times G \to G$ where G is a compact Lie group. If we embed G in a matrix group, then m defines a non-bijective canonical transformation which is quadratic. We shall adopt the following notational conventions. Let $\Delta_M^p := d\delta + \delta d$ be the Laplace–Beltrami operator acting on the space of smooth p forms $C^\infty(\Lambda^p(M))$ on a compact smooth closed Riemannian manifold M of dimension m. We summarize briefly the following well known facts which we shall need, see, for example [2] for further details. Denote the distinct eigenvalues and associated eigenspaces by:

$$\mathrm{Spec}(\Delta_M^p) = \{0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots\},$$
$$E_\lambda(\Delta_M^p) = \left\{\phi \in C^\infty(\Lambda^p(M)) : \Delta_M^p\phi = \lambda\phi\right\}.$$

The spectral multiplicities $\dim\{E_\lambda(\Delta_M^p)\}$ are all finite. Furthermore there is a complete orthonormal decomposition

$$L^2(\Lambda^p(M)) = \bigoplus_{\lambda\in\mathrm{Spec}(\Delta_M^p)} E_\lambda(\Delta_M^p).$$

Let G be a compact connected Lie group which is equipped with a bi-invariant Riemannian metric $\mathrm{d}s_G^2$. Normalize the product metric on $G \times G$ by taking

$$\mathrm{d}s_{G\times G}^2 = 2(\mathrm{d}s_G^2 \oplus \mathrm{d}s_G^2). \tag{1.a}$$

The situation on 0-forms is particularly simple; we shall show in Section 2 that the pull-back of an eigenfunction is again an eigenfunction with the same eigenvalue:

Theorem 1.1. *Let $\mathrm{d}s_G^2$ be a bi-invariant metric on a compact Lie group G. Let $\mathrm{d}s_{G\times G}^2 = 2(\mathrm{d}s_G^2 \oplus \mathrm{d}s_G^2)$. Then the multiplication map $m : G \times G \to G$ is a Riemannian submersion with totally geodesic fibers and $m^*\{E_\lambda(\Delta_G^0)\} \subset E_\lambda(\Delta_{G\times G}^0)$.*

Let π_λ be orthogonal projection on $E_\lambda(\Delta_M^p)$. If $\phi \in C^\infty(\Lambda^p(M))$, let $\mu(\phi)$ be the number of eigenvalues λ so that $\pi_\lambda\phi \neq 0$; this is the number of distinct eigenvalues which are involved in the Fourier series decomposition of ϕ. We shall use the Peter–Weyl theorem in Section 3 to show that:

Theorem 1.2. *Let $\mathrm{d}s_G^2$ be a bi-invariant metric on a compact Lie group G. Let $\mathrm{d}s_{G\times G}^2 = 2(\mathrm{d}s_G^2 \oplus \mathrm{d}s_G^2)$. If $\phi \in E_\lambda(\Delta_G^p)$, then $\mu(m^*\phi) \leq \binom{2\dim\{G\}}{p}\dim\{E_\lambda(\Delta_G^p)\}$.*

The geometry of left invariant 1-forms plays a central role in our discussions. The following result will be established in Section 4:

Theorem 1.3. *Let $\mathrm{d}s_G^2$ be a bi-invariant metric on a compact Lie group G. Let $\mathrm{d}s_{G\times G}^2 = 2(\mathrm{d}s_G^2 \oplus \mathrm{d}s_G^2)$. Let $\phi \in E_\lambda(\Delta_G^1)$ be left invariant. Then one may decompose $m^*\phi = \Phi_1 + \Phi_2$ where $0 \neq \Phi_1 \in E_{\frac{3}{2}\lambda}(\Delta_{G\times G}^1)$ and $0 \neq \Phi_2 \in E_{\frac{1}{2}\lambda}(\Delta_{G\times G}^1)$.*

Theorem 1.2 shows that the pull-back of an eigenform has a finite Fourier series. In Section 5, we will use Theorem 1.3 to establish following result which shows that the number of eigenvalues involved in the Fourier decomposition of $m^*\phi$ can be arbitrarily large:

Theorem 1.4. *Let $p \geq 1$ and let $\mu_0 \in \mathbb{N}$ be given. There exists a bi-invariant metric on a compact Lie group G, there exists λ, and there exists $0 \neq \phi \in E_\lambda(\Delta_G^p)$ so that $\mu(m^*\phi) = \mu_0$.*

The Hodge–DeRham theorem identifies the nth cohomology group $H^n(M; \mathbb{C})$ of M with the space of harmonic n-forms $E_0(\Delta_M^n)$ if M is a compact Riemannian manifold. Thus the eigenvalue 0 has a particular significance. Let

$\Lambda(E_0(\Delta^1_G))$ be the subring generated over $\mathbb{C}$ by the harmonic 1-forms; one has that $\phi \in \Lambda^n(E_0(\Delta^1_G))$ if and only if one can express:

$$\phi = \sum_{|I|=n} a_I \phi^{i_1} \wedge \cdots \wedge \phi^{i_n} \quad \text{where } a_I \in \mathbb{C} \text{ and } \phi^i \in E_0(\Delta^1_G).$$

Theorem 1.5. *Let* $\mathrm{d}s^2_G$ *be a bi-invariant metric on a compact Lie group* G. *Let* $\mathrm{d}s^2_{G\times G} = 2(\mathrm{d}s^2_G \oplus \mathrm{d}s^2_G)$. *Assume* G *connected.*

(1) $\Lambda^n(E_0(\Delta^1_G)) \subset E_0(\Delta^n_G)$.
(2) $\phi \in \Lambda^n(E_0(\Delta^1_G))$ *if and only if* $m^*\phi \in E_0(\Delta^n_{G\times G})$.
(3) *Let* G *be simply connected. If* $\phi \in E_0(\Delta^n_G)$ *for* $n > 0$, $m^*\phi \notin E_0(\Delta^n_{G\times G})$.

One can consider more generally the situation where G and $G \times G$ are endowed with arbitrary left invariant metrics $\mathrm{d}s^2_G$ and $\mathrm{d}s^2_{G\times G}$ where there is no a priori relation assumed between these metrics. The question of when this is a Riemannian submersion is an interesting one and will be studied in more detail in a subsequent paper. For the moment, however, we content ourselves in Section 7 by generalizing Theorem 1.2 to this setting:

Theorem 1.6. *Let* G *and* $G \times G$ *be equipped with left invariant metrics* $\mathrm{d}s^2_G$ *and* $\mathrm{d}s^2_{G\times G}$. *If* $\phi \in E_\lambda(\Delta^p_G)$, *then*

$$\mu(m^*\phi) \le \binom{2\dim\{G\}}{p}^2 \binom{\dim\{G\}}{p}^2 \dim\{E_\lambda(\Delta^p_G)\}^4.$$

We remark that this bound is much worse than the bound given in Theorem 1.2; at two different points in the proof we shall need to pass from a left invariant subspace to a bi-invariant subspace and this greatly increases estimate on the dimension.

2. The geometry of the multiplication map *m*

Let $\pi : X \to Y$ be a surjective smooth map where X and Y are compact Riemannian manifolds. We suppose that π is a submersion, i.e. that the map $\pi_* : T_x X \to T_{\pi x} Y$ is surjective for every $x \in X$, and let $\mathcal{V}$ (resp. $\mathcal{H}$) be the associated vertical (resp. horizontal) distribution:

$$\mathcal{V} := \{\xi \in TX : \pi_*\xi = 0\} \quad \text{and} \quad \mathcal{H} := \mathcal{V}^\perp.$$

We say that π is a *Riemannian submersion* if $\pi_* : \mathcal{H}_x \to TY_{\pi x}$ is an isometry $\forall x$.

The following example is instructive. Let $m(u, v) = u + v$ define a linear map from $\mathbb{R}^{2n} \to \mathbb{R}^n$. Take the standard Euclidean metric on $\mathbb{R}^n$. We may identify $T_x\mathbb{R}^{2n} = \mathbb{R}^{2n}$ and $T_y\mathbb{R}^n = \mathbb{R}^n$. Under this identification,

$$\mathcal{V} = \operatorname*{Span}_{\xi\in\mathbb{R}^n}\left\{\left(\frac{1}{2}\xi, -\frac{1}{2}\xi\right)\right\} \quad \text{and} \quad \mathcal{H} = \operatorname*{Span}_{\xi\in\mathbb{R}^n}\left\{\left(\frac{1}{2}\xi, \frac{1}{2}\xi\right)\right\}.$$

We have $m_*(\frac{1}{2}\xi, \frac{1}{2}\xi) = \xi$. Thus if ξ is a unit vector in $T_y\mathbb{R}^n$, we need that $(\frac{1}{2}\xi, \frac{1}{2}\xi)$ is a unit vector in $T_x\mathbb{R}^{2n}$. This motivates the factor of 2 which appears in Eq. (1.a) since the ordinary Euclidean length of $(\frac{1}{2}\xi, \frac{1}{2}\xi)$ would be $\frac{1}{2}$ and not 1. With this normalization, m becomes a Riemannian submersion.

More generally, let G be a Lie group which is equipped with a bi-invariant Riemannian metric $\mathrm{d}s^2_G$. Let $m(x, y) = xy$ be the multiplication operator from $G \times G \to G$. Let $\{e^L_i\}$ (resp. $\{e^R_i\}$) be an orthonormal frame of left (resp. right) invariant vector fields on G. We assume $e^L_i(1) = e^R_i(1) = e_i$ where $1 \in G$ is the unit of the group and where $\{e_i\}$ is an orthonormal basis for $T_1(G)$. Let exp be the exponential map in the group. Then the flows Ξ^L_i and Ξ^R_i of these vector fields are:

$$\Xi^L_i : (g, t) \to g\exp(te_i) \quad \text{and} \quad \Xi^R_i : (g, t) \to \exp(te_i)g.$$

The multiplication map m defines a smooth surjective map $m : G \times G \to G$. Consider the following curves in $G \times G$ with initial position (g_1, g_2):

$$\gamma_i^{g_1,g_2} : t \to \left(g_1 \exp\left(\frac{1}{2}te_i\right), \exp\left(-\frac{1}{2}te_i\right) g_2\right),$$
$$\varrho_i^{g_1,g_2} : t \to \left(g_1 \exp\left(\frac{1}{2}te_i\right), \exp\left(\frac{1}{2}te_i\right) g_2\right),$$
$$\tau_i^{g_1,g_2} : t \to (\exp(te_i)g_1, g_2).$$

We may identify $T(G \times G) = TG \oplus TG$. Because $m\tau_i^{g_1,g_2} : t \to \exp(te_i)g_1 g_2$,

$$m_*\{\dot{\tau}_i^{g_1,g_2}(0)\} = e_i^R(m(g_1, g_2)).$$

Consequently m_* is surjective so m is a submersion. As $m\gamma_i^{g_1,g_2} : t \to g_1 g_2$ is independent of t, one has $\dot{\gamma}_i = \frac{1}{2}(e_i^L, -e_i^R) \in \ker\{m_*\}$. It now follows that

$$\begin{aligned} \mathcal{V} &:= \ker\{m_*\} = \text{Span}\left\{V_i := \frac{1}{2}(e_i^L, -e_i^R)\right\}, \\ \mathcal{H} &:= \ker\{m_*\}^\perp = \text{Span}\left\{H_i := \frac{1}{2}(e_i^L, e_i^R)\right\}. \end{aligned} \tag{2.a}$$

Let L_g and R_g denote left and right multiplication in the group. As $\dot{\varrho}_i = H_i$,

$$m_{*(g_1,g_2)}\{H_i(g_1, g_2)\} = (L_{g_1})_*(R_{g_2})_* e_i. \tag{2.b}$$

Since L_{g_1} and R_{g_2} are isometries, it follows that $\{m_* H_i(g_1, g_2)\}$ is an orthonormal basis for $T_{g_1 g_2}G$. We have defined $\mathrm{d}s^2_{G\times G} = 2(\mathrm{d}s^2_G \oplus \mathrm{d}s^2_G)$. We show that m is a Riemannian submersion by computing:

$$(H_i, H_j)_{G\times G} = 2\frac{1}{4}\left\{(e_i^L, e_j^L)_G + (e_i^R, e_j^R)_G\right\} = \delta_{ij},$$
$$(H_i, V_j)_{G\times G} = 2\frac{1}{4}\left\{(e_i^L, e_j^L)_G - (e_i^R, e_j^R)_G\right\} = 0,$$
$$(V_i, V_j)_{G\times G} = 2\frac{1}{4}\left\{(e_i^L, e_j^L)_G + (e_i^R, e_j^R)_G\right\} = \delta_{ij}.$$

Fix $h \in G$. The map $T_h : (g_1, g_2) \to (hg_2^{-1}, g_1^{-1}h)$ is an isometry $G \times G$. Clearly $T_h(g_1, g_2) = (g_1, g_2)$ if and only if $g_1 = hg_2^{-1}$ and $g_2 = g_1^{-1}h$ or equivalently if $g_1 g_2 = h$. Thus the fixed point set of T is $m^{-1}(h)$. Since the fixed point set of an isometry consists of the disjoint union of totally geodesic submanifolds, the fibers of m, which are connected submanifolds diffeomorphic to G, are totally geodesic. It now follows that the mean curvature covector vanishes. Theorem 4.3.1 of [3] shows $m^*\Delta_G^0 = \Delta_{G\times G}^0 m^*$. This completes the proof of Theorem 1.1.

3. The Peter–Weyl theorem

We recall the classical Peter–Weyl theorem; for further details see, for example, [5,6]. Let G be a compact Lie group which is equipped with a bi-invariant metric; assume the metric is normalized so G has unit volume. If ρ is a smooth left representation of G on a finite dimensional complex vector space V, then by averaging an arbitrary inner product on V over the group we can always choose an inner product on V which is preserved by ρ. Thus any such representation is unitarizable. Let $\text{Irr}(G)$ be the set of isomorphism classes of finite dimensional irreducible unitary left representations of G. We can decompose any finite dimensional left representation space V as a direct sum of irreducibles:

$$V = \bigoplus_{\rho\in\text{Irr}(G)} n_\rho V_\rho;$$

the multiplicities n_ρ are independent of the particular decomposition chosen and are non-zero for only finitely many ρ.

Let $\{e_i\}$ be an orthonormal basis for V_ρ where $\rho \in \mathrm{Irr}(G)$. We may expand $\rho(g)e_i = \sum_j \rho_{ij}(g)e_j$; the functions $\rho_{ij} \in C^\infty(G)$ are said to be the matrix coefficients of ρ. We let

$$H_\rho := \operatorname*{Span}_{1\le i,j\le \dim(\rho)} \{\rho_{ij}\} \subset L^2(G).$$

It is easily verified that H_ρ is invariant under both the left and right group action and that H_ρ is independent of the particular orthonormal basis chosen for V_ρ; isomorphic representations determine the same space. Furthermore, as a left representation space for G, H_ρ is isomorphic to $\dim(\rho)$ copies of the original representation ρ.

If V is any finite dimensional subspace of $L^2(G)$ which is left invariant under G and which is abstractly isomorphic to V_ρ as a representation space, then one has $V \subset H_\rho$; to put it another way, H_ρ contains all the left submodules of $L^2(G)$ which are isomorphic to V_ρ. Furthermore, we have a complete orthogonal direct sum decomposition

$$L^2(G) = \bigoplus_{\rho\in\mathrm{Irr}(G)} H_\rho = \bigoplus_{\rho\in\mathrm{Irr}(G)} \dim(\rho)\cdot V_\rho.$$

This means that $\{\rho_{ij}\}_{1\le i,j\le\dim(\rho),\rho\in\mathrm{Irr}(G)}$ is a complete orthonormal basis for $L^2(G)$.

More generally, let $\{\phi_L^i\}$ be an orthonormal basis for the space of left invariant 1-forms. If one has that $I = \{1 \le i_1 < \cdots < i_p \le \dim(G)\}$ is a multi-index, let $\Phi_L^I := \phi_L^{i_1} \wedge \cdots \wedge \phi_L^{i_p}$; the Φ_L^I are an orthonormal basis for the space of left invariant p-forms and as a left representation space for G one has:

$$L^2(\Lambda^p(G)) = \bigoplus_{\rho\in\mathrm{Irr}(G),|I|=p} H_\rho \otimes \Phi_L^I = \bigoplus_{\rho\in\mathrm{Irr}(G)} \binom{\dim\{G\}}{p} \dim(V_\rho)V_\rho. \tag{3.a}$$

The subspace $H_\rho^p := \oplus_{|I|=p} H_\rho \cdot \Phi_L^I$ is a bi-invariant G submodule of $L^2(\Lambda^p(G))$ which contain every left subrepresentation of G on $L^2(\Lambda^p(G))$ isomorphic to V_ρ.

Let π_λ be orthogonal projection from $L^2(\Lambda^p(G))$ to $E_\lambda(\Delta_G^p)$ and let $\mu(\phi)$ be the number of eigenvalues λ so $\pi_\lambda(\phi) \neq 0$. We prepare for the proof of Theorem 1.2 by establishing:

Lemma 3.1. *Let $H \subset L^2(\Lambda^p(G))$ be invariant under the action of L_g for all $g \in G$. If $\phi \in H$, then $\mu(\phi) \le \binom{\dim\{G\}}{p} \dim\{H\}$.*

Proof. Clearly $\pi_\lambda H$ is non-trivial if and only if there exists $\rho \in \mathrm{Irr}(G)$ so that the multiplicities satisfy:

$$n_H(\rho) > 0 \quad \text{and} \quad n_{E_\lambda(\Delta_G^p)}(\rho) > 0.$$

Note that only a finite number of representations appear in H and only a finite number of eigenspaces involve any given representation. By Eq. (3.a),

$$\begin{aligned}\mu(\phi) &\le \sum_{\rho\in\mathrm{Irr}(G):n_\rho(H)\neq 0} \left\{ \sum_{\lambda:n_\rho(E_\lambda(\Delta_G^p))\neq 0} 1 \right\} \\ &\le \sum_{\rho\in\mathrm{Irr}(G):n_\rho(H)\neq 0} \left\{ \binom{\dim\{G\}}{p} \dim\{V_\rho\} \right\} \le \binom{\dim\{G\}}{p} \dim\{H\}. \quad \square\end{aligned}$$

We can now establish Theorem 1.2. It is convenient to introduce $\tilde m(g,h) = gh^{-1}$. Let $H = E_\lambda(\Delta_G^p)$. Since the metric is bi-invariant, the Laplacian and hence the eigenspaces are preserved by both left and right multiplication. Let $\tilde H := \tilde m^* H$. We compute:

$$\tilde m\{L_{g,h}^{G\times G}(a_1,a_2)\} = \tilde m(ga_1, ha_2) = ga_1a_2^{-1}h^{-1} = L_g^G R_{h^{-1}}^G \tilde m(a_1,a_2),$$
$$\{L_{g,h}^{G\times G}\}^* \tilde m^* = \tilde m^*(R_{h^{-1}}^G)^*(L_g^G)^*.$$

Since H is invariant under both the left and right actions of G, $\tilde H$ is invariant under the left action of $G \times G$. We replace the group in question by $G \times G$ and apply Lemma 3.1 to estimate $\mu(\tilde m^*\phi)$. Since the metric on $G \times G$ is bi-invariant, $\psi(x, y) := (x, y^{-1})$ is an isometry of $G \times G$. We have

$$a_1 a_2 = m(a_1, a_2) = a_1(a_2^{-1})^{-1} = \tilde{m}(\psi(a_1, a_2))$$

and thus $m^* = \psi^* \tilde{m}^*$. Consequently $\mu(m^*\phi) = \mu(\psi^*\tilde{m}^*\phi) = \mu(\tilde{m}^*\phi)$. □

4. Left invariant 1-forms

Let $\Lambda_L^p(G)$ be the finite dimensional vector space of left invariant p-forms on G. Define the left and right actions of G on $G \times G$ by:

$$\begin{aligned} &L_{1,g} : (x, y) \to (gx, y), \qquad L_{2,g} : (x, y) \to (x, gy), \\ &R_{1,g} : (x, y) \to (xg, y), \qquad R_{2,g} : (x, y) \to (x, yg). \end{aligned} \tag{4.a}$$

Consider the following subspaces:

$$\tilde{\Lambda}^p(G \times G) = \{\theta \in C^\infty(\Lambda^p(G \times G)) : L_{1,g}^*\theta = \theta, R_{1,g^{-1}}^* L_{2,g}^*\theta = \theta \ \forall g \in G\}.$$

Lemma 4.1. *Adopt the notation established above. Then:*

(1) $d_{G\times G}\{\tilde{\Lambda}^p(G \times G)\} \subset \tilde{\Lambda}^{p+1}(G \times G)$, $\delta_{G\times G}\{\tilde{\Lambda}^{p+1}(G \times G)\} \subset \tilde{\Lambda}^p(G \times G)$, $\Delta_{G\times G}^p\{\tilde{\Lambda}^p(G \times G)\} \subset \tilde{\Lambda}^p(G \times G)$, *and* $\tilde{\Lambda}^p(G \times G) \wedge \tilde{\Lambda}^q(G \times G) \subset \tilde{\Lambda}^{p+q}(G \times G)$.
(2) *The map* $\theta \to \theta(1)$ *is an isomorphism from* $\tilde{\Lambda}^p(G \times G)$ *to* $\Lambda^p(G \times G)(1)$.
(3) $m^*\{\Lambda_L^p(G)\} \subset \tilde{\Lambda}^p(G \times G)$.

Proof. Assertion (1) follows since the maps of Eq. (4.a) are isometries and thus the pull-backs defined by these maps commute with d, δ, Δ, and $\wedge$. To prove Assertion (2), define an action A of $G \times G$ on $G \times G$ by setting:

$$A_{g,h} : (a, b) \to (gah^{-1}, hb)$$

this is a fixed point free transitive isometric group action since

$$A_{g_1,h_1} A_{g_2,h_2} = A_{g_1 g_2, h_1 h_2}.$$

This exhibits $G \times G$ as a homogeneous space. We have furthermore that:

$$\begin{aligned} &m(ga, b) = gm(a, b), \qquad m \circ L_{1,g} = L_g \circ m, \qquad m^* L_g^* = L_{1,g}^* m^*, \\ &m(ag^{-1}, gb) = m(a, b), \qquad m \circ L_{2,g} R_{1,g^{-1}} = m, \qquad R_{1,g^{-1}}^* L_{2,g}^* m^* = m^*. \end{aligned}$$

Suppose that $\phi \in \Lambda_L^p(G)$. Then $L_g^*\phi = \phi$ for all g. Consequently

$$L_{1,g}^* m^*\phi = m^* L_g^*\phi = m^*\phi \quad \text{and} \quad R_{1,g^{-1}}^* L_{2,g}^* m^*\phi = m^*\phi.$$

Assertion (3) follows. □

Fix an orthonormal frame $\{\phi_L^i\}$ for $\Lambda_L^1(G)$ so that

$$\Delta_G^1\{\phi_L^i\} = \lambda_i \phi_L^i. \tag{4.b}$$

Since right and left multiplication commute, right multiplication preserves $\Lambda_L^1(G)$. Thus we may decompose

$$R_g^*\phi_L^i = \sum_j \xi_{ij}(g)\phi_L^j. \tag{4.c}$$

Since $R_g R_h = R_{hg}$ and since $R_1 = \text{id}$, we have

$$\xi_{ij}(g)\xi_{jk}(h) = \xi_{ik}(hg) \quad \text{and} \quad \xi_{ij}(1) = \delta_{ij}.$$

We may decompose $\Lambda^1(G \times G) = \Lambda^1(G) \oplus \Lambda^1(G)$. Define

$$\Phi_1^i(u, v) = \sum_j \xi_{ij}(v)\phi_L^j(u) \oplus 0 \quad \text{and} \quad \Phi_2^i(u, v) = 0 \oplus \phi_L^i(v).$$

Lemma 4.2. *Adopt the notation established above.*

(1) $\{\Phi_1^i, \Phi_2^i\}$ *is a basis for* $\tilde{\Lambda}^1(G \times G)$.
(2) $m^*\phi_L^i = \Phi_1^i + \Phi_2^i$.
(3) $\Delta^1_{G\times G}\Phi_1^i = \frac{3}{2}\lambda_i\Phi_1^i$ *and* $\Delta^1_{G\times G}\Phi_2^i = \frac{1}{2}\lambda_i\Phi_2^i$.

Proof. It is immediate from the definition that $L^*_{1,g}\Phi_1^i = \Phi_1^i$, $L^*_{1,g}\Phi_2^i = \Phi_2^i$, and $R^*_{1,g^{-1}}L^*_{2,g}\Phi_2^i = \Phi_2^i$. We use Eq. (4.c) to see:

$$\begin{aligned}\{R^*_{1,g^{-1}}L^*_{2,g}\Phi_1^i\}(u,v) &= \sum_{jk}\xi_{ij}(gv)\xi_{jk}(g^{-1})\phi_L^k(u)\oplus 0\\ &= \sum_{jkl}\xi_{il}(v)\xi_{lj}(g)\xi_{jk}(g^{-1})\phi_L^k(u)\oplus 0\\ &= \sum_k \xi_{ik}(v)\phi_L^k(u)\oplus 0 = \Phi_1^i(u,v).\end{aligned}$$

Thus $\Phi_1^i \in \tilde{\Lambda}^1(G\times G)$ and $\Phi_2^i \in \tilde{\Lambda}^1(G\times G)$. Because $\Phi_1^i(1,1) = \phi_L^i(1)\oplus 0$ and because $\Phi_2^i(1,1) = 0\oplus\phi_L^i(1)$, Assertion (1) now follows from Assertion (2) of Lemma 4.1. We dualize Eqs. (2.a) and (2.b) to see that

$$\{m^*\phi_L^i\}(1,1) = \phi_L^i(1)\oplus\phi_L^i(1) = \{\Phi_1^i + \Phi_2^i\}(1,1).$$

The identity of Assertion (2) of Lemma 4.2 now follows from Assertion (1) of Lemma 4.2 and from Assertion (3) of Lemma 4.1.

Suppose $\phi \in \Lambda^1_L(G)$. Then $\delta_G\phi \in \Lambda^0_L(G)$ is left invariant and hence $\delta_G\phi = c$ is constant. Since $dc = 0$,

$$c^2\,\mathrm{vol}(G) = (\delta_G\phi, \delta_G\phi)_{L^2(G)} = (\phi, d_G\delta_G\phi)_{L^2(\Lambda^1 G)} = 0.$$

Similarly if $\Phi \in \tilde{\Lambda}^1(G\times G)$, then $\delta_{G\times G}\Phi \in \tilde{\Lambda}^0(G\times G)$ is invariant under the transitive group action A defined above. Consequently $\delta_{G\times G}\Phi = C$ constant and again

$$C^2\,\mathrm{vol}(G\times G) = (\delta_{G\times G}\Phi, \delta_{G\times G}\Phi)_{L^2(G\times G)} = (\Phi, d_{G\times G}\delta_{G\times G}\Phi)_{L^2\Lambda^1(G\times G)} = 0.$$

Consequently one may express:

$$\Delta^1_G\{\phi_L^i\} = \delta_G d_G\{\phi_L^i\} \quad\text{and}\quad \Delta^1_{G\times G}\{\Phi_a^i\} = \delta_{G\times G}d_{G\times G}\{\Phi_a^i\} \quad \text{for } a = 1,2. \tag{4.d}$$

Decompose

$$d_G\{\phi_L^i\} = \sum_{j<k} C_{ijk}\phi_L^j\wedge\phi_L^k \quad\text{and}\quad \delta_G\{\phi_L^j\wedge\phi_L^k\} = \sum_i D_{ijk}\phi_L^i.$$

We compute:

$$\begin{aligned}D_{ijk}\mathrm{vol}(G) &= (\delta_G\{\phi_L^j\wedge\phi_L^k\}, \phi_L^i)_{L^2(\Lambda^1 G)} = (\phi_L^j\wedge\phi_L^k, d\phi_L^i)_{L^2(\Lambda^2 G)}\\ &= C_{ijk}\mathrm{vol}(G).\end{aligned}$$

Consequently $D_{ijk} = C_{ijk}$. Eqs. (4.b) and (4.d) yield:

$$\sum_{j<k,l} C_{ljk}C_{ijk}\phi_L^l = \delta_G\left\{\sum_{j<k} C_{ijk}\phi_L^j\wedge\phi_L^k\right\} = \delta_G d_G\{\phi_L^i\} = \Delta^1_G\{\phi_L^i\} = \lambda_i\phi_L^i$$

and consequently

$$\sum_{j<k} C_{ljk}C_{ijk} = \lambda_i\delta^{il}. \tag{4.e}$$

Let $\sigma_2(g_1, g_2) = g_2$ denote projection on the second factor. Since $\Phi_2^i = \sigma_2^* \phi_L^i$ and since $\Phi_1^i + \Phi_2^i = m^* \phi_L^i$,

$$\begin{aligned} d_{G\times G}\{\Phi_2^i\} &= \sum_{j<k} C_{ijk} \Phi_2^j \wedge \Phi_2^k, \\ d_{G\times G}\{\Phi_1^i + \Phi_2^i\} &= \sum_{j<k} C_{ijk} (\Phi_1^j + \Phi_2^j) \wedge (\Phi_1^k + \Phi_2^k), \\ d_{G\times G}\{\Phi_1^i\} &= d_{G\times G}\{\Phi_1^i + \Phi_2^i\} - d_{G\times G}\{\Phi_2^i\} \\ &= \sum_{j<k} C_{ijk} \left\{ \Phi_1^j \wedge \Phi_1^k + \Phi_1^j \wedge \Phi_2^k + \Phi_2^j \wedge \Phi_1^k \right\}. \end{aligned} \tag{4.f}$$

We expand $\delta_{G\times G}\{\Phi_2^j \wedge \Phi_2^k\} = \sum_i \{D_{1,ijk} \Phi_1^i + D_{2,ijk} \Phi_2^i\}$. Then, taking into account the normalizing factor of 2 in Eq. (1.a) which dually yields a factor of $\frac{1}{2}$ on the inner product for $\Lambda^1(G \times G)$ and a factor of $\frac{1}{4}$ on the inner product for $\Lambda^2(G \times G)$, one has:

$$\begin{aligned} \frac{1}{2} D_{1,ijk} \mathrm{vol}(G \times G) &= (\delta_{G\times G}\{\Phi_2^j \wedge \Phi_2^k\}, \Phi_1^i)_{L^2(\Lambda^1(G\times G))} \\ &= (\Phi_2^j \wedge \Phi_2^k, d_{G\times G}\{\Phi_1^i\})_{L^2(\Lambda^2(G\times G))} = 0, \\ \frac{1}{2} D_{2,ijk} \mathrm{vol}(G \times G) &= (\delta_{G\times G}\{\Phi_2^j \wedge \Phi_2^k\}, \Phi_2^i)_{L^2(\Lambda^1(G\times G))} \\ &= (\Phi_2^j \wedge \Phi_2^k, d_{G\times G}\{\Phi_2^i\})_{L^2(\Lambda^2(G\times G))} = \frac{1}{4} C_{ijk} \mathrm{vol}(G \times G). \end{aligned}$$

This shows that

$$D_{1,ijk} = 0 \quad \text{and} \quad D_{2,ijk} = \frac{1}{2} C_{ijk}. \tag{4.g}$$

Eqs. (4.d)–(4.g) yield:

$$\Delta_{G\times G}(\Phi_2^i) = \delta_{G\times G} d_{G\times G}\{\Phi_2^i\} = \frac{1}{2} \sum_{l,j<k} C_{ljk} C_{ijk} \Phi_2^l = \frac{1}{2} \lambda_i \Phi_2^i.$$

Similarly

$$\delta_{G\times G}\{\Phi_1^j \wedge \Phi_1^k\} = \delta_{G\times G}\{\Phi_2^j \wedge \Phi_1^k\} = \delta_{G\times G}\{\Phi_1^j \wedge \Phi_2^k\} = \frac{1}{2} \sum_l C_{ljk} \Phi_1^l$$

and thus $\Delta^1_{G\times G}\{\Phi_1^i\} = \frac{3}{2} \lambda_i \Phi_1^i$. □

5. Eigenforms whose pull-back has many non-zero Fourier coefficients

Let S^3 be the unit sphere in the quaternions $\mathbb{H} = \mathbb{R}^4$; this is a compact connected Lie group and the standard round metric is the only bi-invariant metric on S^3 modulo rescaling. Fix

$$0 \neq f \in E_{\lambda_0}(\Delta^0_{S^3})$$

with $\lambda_0 \neq 0$. Since the first cohomology group of S^3 is trivial, there are no non-trivial harmonic 1-forms on S^3. Thus we may choose

$$0 \neq \phi \in \Lambda^1_L(S^3) \cap E_{\lambda_1}(\Delta^1_{S^3})$$

for some $\lambda_1 > 0$; we refer to [4,10] for additional details concerning the spectral geometry of S^3; S^3 could be replaced by any non-Abelian compact connected Lie group in this construction.

We first prove Theorem 1.4 in the special case that $p = 1$. Suppose that $\mu_0 = 2k$. Choose real numbers $0 < t_1 < \cdots < t_k < 1$. Choose $s_1 > \cdots > s_k > 1$ so

$$s_\alpha \lambda_0 + t_\alpha \lambda_1 = \lambda_0 + \lambda_1 \quad \text{for } 1 \leq \alpha \leq k.$$

Let G_α be S^3 with the rescaled metric $\mathrm{d}s^2_{G_\alpha} := t_\alpha^{-1}\mathrm{d}s^2_{S^3}$ and let $\phi^\alpha = \phi \in \Lambda^1_L(G_\alpha)$. Let $\bar{G}_\alpha$ be S^3 with the rescaled metric $\mathrm{d}s^2_{\bar{G}_\alpha} := s_\alpha^{-1}\mathrm{d}s^2_{S^3}$ and let $f_\alpha = f \in C^\infty(\bar{G}_\alpha)$. After taking into account the effect of the rescaling, we have

$$f_\alpha \in E_{s_\alpha\lambda_0}(\Delta^0_{\bar{G}_\alpha}), \qquad \mathrm{d}f_\alpha \in E_{s_\alpha\lambda_0}(\Delta^1_{\bar{G}_\alpha}), \quad \text{and} \quad \phi^\alpha \in E_{t_\alpha\lambda_1}(\Delta^1_{G_\alpha}).$$

Let $G = G_1 \times \cdots \times G_k \times \bar{G}_1 \times \cdots \times \bar{G}_k$. Decompose $m^*(\phi^\alpha) = \Phi^\alpha_1 + \Phi^\alpha_2$. Let $\psi := \sum_\alpha f_\alpha\phi^\alpha$. As the structures decouple, one has:

$$\Delta^1_G\{\psi\} = \sum_\alpha (s_\alpha\lambda_0 + t_\alpha\lambda_1) f_\alpha\phi^\alpha = (\lambda_0 + \lambda_1)\psi.$$

We can apply Theorem 1.3 to see

$$\begin{aligned}\Delta^1_{G\times G} m^*\psi &= \sum_\alpha \left\{\left(s_\alpha\lambda_0 + \frac{3}{2}t_\alpha\lambda_1\right) m^* f_\alpha \cdot \Phi^\alpha_1 + \left(s_\alpha\lambda_0 + \frac{1}{2}t_\alpha\lambda_1\right) m^* f_\alpha \cdot \Phi^\alpha_2\right\} \\ &= \sum_\alpha \left\{\left(\lambda_0 + \lambda_1 + \frac{1}{2}t_\alpha\lambda_1\right) m^* f_\alpha \cdot \Phi^\alpha_1 + \left(\lambda_0 + \lambda_1 - \frac{1}{2}t_\alpha\lambda_1\right) m^* f_\alpha \cdot \Phi^\alpha_2\right\}.\end{aligned}$$

The computations performed above then yield $\psi \in E_{\lambda_0+\lambda_1}(\Delta^1_G)$. Furthermore:

$$\begin{aligned}&m^*(f_\alpha)\Phi^\alpha_1 \in E_{\lambda_0+\lambda_1+\frac{1}{2}t_\alpha\lambda_1}(\Delta^1_{G\times G}),\\ &m^*(f_\alpha)\Phi^\alpha_2 \in E_{\lambda_0+\lambda_1-\frac{1}{2}t_\alpha\lambda_1}(\Delta^1_{G\times G}).\end{aligned}$$

Since $0 < t_1 < \cdots < t_k$, $m^*\psi$ has a Fourier decomposition which involves $2k = \mu_0$ distinct eigenvalues. This establishes Theorem 1.4 if $p = 1$ and if μ_0 is even.

If $\mu_0 = 2k + 1$ is odd, we choose s_0 so $s_0\lambda_0 = \lambda_0 + \lambda_1$. Then $f_0 \in E_{\lambda_1+\lambda_2}(\Delta^0_{\bar{G}_0})$. We apply the construction described above to $G = G_1 \times \cdots \times G_k \times \bar{G}_0 \times \cdots \times \bar{G}_k$ and to $\psi = df_0 + f_1\phi^1 + \cdots + f_k\phi^k$; the latter factors are not present if $\mu_0 = 1$. Since $m^*df_0 \in E_{\lambda_0+\lambda_1}(\Delta^1_{G\times G})$, there are $2k + 1$ distinct eigenvalues which are involved in the Fourier decomposition of ψ. This completes the proof of Theorem 1.4 if $p = 1$. We take the product of G with circles S^1 and replace ϕ by $\phi \wedge d\theta_1 \wedge \cdots \wedge d\theta_p$, where θ_β is the usual periodic parameter on S^1, to complete the proof if $p \geq 1$. □

6. Harmonic forms

Before beginning the proof of Theorem 1.5, we must establish some technical results. Let $\mathfrak{g}_L$ be the Lie algebra of left invariant vector fields on G. The following results are well known; we sketch the proofs briefly:

Lemma 6.1. *Let $\mathrm{d}s^2_G$ be a bi-invariant metric on a compact connected Lie group G.*

(1) *If $\theta \in E_0(\Delta^n_G)$, then θ is bi-invariant.*
(2) *If η is a bi-invariant vector field, then $\nabla\eta = 0$.*
(3) *Let $\theta \in \Lambda^1_L(G)$. If $d\theta = 0$, then $\nabla\theta = 0$.*
(4) *If $\Theta \in \Lambda^n(E_0(\Delta^1_G))$, then $\nabla\Theta = 0$ and $\Theta \in E_0(\Delta^n_G)$.*

Proof. The Hodge–DeRham theorem provides a natural identification of $E_0(\Delta^n_G)$ with the cohomology group $H^n(G; \mathbb{C})$. In particular, this identification is compatible with the action of L^*_g and R^*_g. Since G is connected, L^*_g and R^*_g act trivially on $H^n(G; \mathbb{C})$ and hence on $E_0(\Delta^n_G)$. Assertion (1) follows.

To prove Assertion (2), we use well known facts concerning bi-invariant metrics on Lie groups; see, for example, [9]. Let $\exp(t\xi)$ be the integral curve through the identity for $\xi \in \mathfrak{g}_L(G)$. Let η be bi-invariant. Assertion (2) follows as:

$$\nabla_\xi\eta = \frac{1}{2}[\xi, \eta] = \frac{1}{2}\partial_t\left\{(L_{\exp(t\xi)})_*(R_{\exp(-t\xi)})_*\eta\right\}|_{t=0} = \partial_t\eta|_{t=0} = 0.$$

Let $\theta \in \Lambda^1_L(G)$ with $d\theta = 0$. Since $\delta\theta$ is left invariant, $\delta\theta = c$ is constant. Since $\Delta^0_G c = 0$, $\delta\theta = 0$. Thus θ is harmonic and hence bi-invariant. We use the metric to raise and lower indices and identify the tangent and cotangent spaces. Let η be the corresponding dual bi-invariant vector field. By Assertion (2), η is parallel. Thus, dually, θ is parallel. This proves Assertion (3).

Let $\Theta \in \Lambda^n(E_0(\Delta_G^1))$. Then there are constants a_I and harmonic 1-forms θ_L^i so

$$\Theta = \sum_{|I|=n} a_I \theta_L^{i_1} \wedge \cdots \wedge \theta_L^{i_n}.$$

By assertion (3), $\nabla\theta_L^i = 0$. Consequently $\nabla\Theta = 0$. On the other hand, one has

$$d + \delta = \sum_i \{\mathrm{ext}(e^i) - \mathrm{int}(e^i)\}\nabla_{e_i}$$

where $\{e_i\}$ and $\{e^i\}$ are dual orthonormal frames for TG and T^*G and where $\mathrm{ext}(\cdot)$ and $\mathrm{int}(\cdot)$ denote exterior and interior multiplication. Thus parallel forms are necessarily harmonic. Assertion (4) follows. $\square$

We distinguish the two factors in the product to decompose

$$\Lambda^n(G \times G) = \oplus_{p+q=n} \Lambda^p(G_1) \otimes \Lambda^q(G_2).$$

We let $\pi_{p,q}$ denote orthogonal projection on the various components. The Künneth formula shows

$$H^n(G \times G; \mathbb{C}) = \oplus_{p+q=n} H^p(G_1; \mathbb{C}) \otimes H^q(G_2; \mathbb{C})$$

and, as we have taken a product metric on $G \times G$, we have a corresponding decomposition in the geometric context:

$$C^\infty(\Lambda^n(G \times G)) = \oplus_{p+q=n} C^\infty\{\Lambda^p(G_1) \otimes \Lambda^q(G_2)\},$$
$$\Delta^n_{G\times G} = \oplus_{p+q=n} \{\Delta^p_{G_1} \otimes \mathrm{id} + \mathrm{id} \otimes \Delta^q_{G_2}\},$$
$$E_0(\Delta^n_{G\times G}) = \oplus_{p+q=n} \{E_0(\Delta^p_{G_1}) \otimes E_0(\Delta^q_{G_2})\}.$$

Assertion (1) of Theorem 1.5 follows from Lemma 6.1. To prove Assertion (2) of Theorem 1.5, suppose

$$\phi = \sum_{|I|=n} a_I \theta^{i_1} \wedge \cdots \wedge \theta^{i_n} \in \Lambda^n(E_0(\Delta_G^1)) \quad \text{for } \theta^j \in E_0(\Delta_G^1).$$

As θ^j is bi-invariant, $\theta^j \oplus \theta^j \in \tilde{\Lambda}^1(G \times G)$. Since $m^*\theta(1,1) = \theta^j(1,1) \oplus \theta^j(1,1)$, $m^*\theta^j = \theta^j \oplus \theta^j$. As $d\theta^j = 0$, $dm^*\theta^j = m^*d\theta^j = 0$. Thus $m^*\theta^j \in E_0(\Delta^1_{G\times G})$ so $m^*\phi \in \Lambda^n(E_0(\Delta^1_{G\times G}))$ is harmonic.

Conversely suppose that $m^*\phi \in E_0(\Delta^n_{G\times G})$. We then have $\pi_{0,n}m^*\phi$ is harmonic. Since $\pi_{0,n}m^*\phi = \sigma_2^*\phi = 0 \oplus \phi$, ϕ is harmonic and hence bi-invariant. Decompose $\phi = \sum_{|I|=n} a_I \phi_L^I$ as a sum of left invariant n-forms where the coefficients a_I are constant. As $m^*\phi$ is harmonic, $m^*\phi$ is left invariant and decomposes in the form:

$$m^*\phi = \sum_{0<i_1<\cdots<i_n<\dim(G)} a_{i_1\ldots i_n}(\phi_L^{i_1} \oplus 0 + 0 \oplus \phi_L^{i_1}) \wedge \cdots \wedge (\phi_L^{i_n} \oplus 0 + 0 \oplus \phi_L^{i_n}).$$

Choose the indexing convention so $\{\phi^1, \ldots, \phi^k\}$ is an orthonormal basis for $E_0(\Delta_G^1)$ and so $\{\phi^{k+1}, \ldots, \phi^{\dim(G)}\}$ completes the set to an orthonormal basis for $\Lambda_L^1(G)$. We suppose that $\phi \notin \Lambda^n(\phi^1, \ldots, \phi^k)$ and argue for a contradiction. Choose a minimal so $a_{i_1,\ldots,i_a,j_1,\ldots,j_b} \neq 0$ where $i_a \le k$ and $k < j_1 < j_2 < \cdots < j_b$. By hypothesis $a < n$ so $b \ge 1$. Let

$$\phi_0 := \phi_L^{i_1} \wedge \cdots \wedge \phi_L^{i_a} \wedge \phi_L^{j_2} \wedge \cdots \wedge \phi_L^{j_b},$$
$$\phi := \tilde{\phi} \wedge \phi_0 + \text{other terms}$$

where the other terms do not involve the monomial ϕ_0 and where $0 \neq \tilde{\phi} \notin E_0(\Delta_G^1)$; $\tilde{\phi} = \mathrm{int}(\phi_0)\phi \in \Lambda_L^1(G)$. We may then expand

$$\pi_{1,n-1}m^*\phi = (\tilde{\phi} \oplus 0) \wedge (0 \oplus \phi_0) + \text{other terms},$$
$$\mathrm{d}\pi_{1,n-1}m^*\phi = (\mathrm{d}\tilde{\phi} \oplus 0) \wedge (0 \oplus \phi_0) + \text{other terms}.$$

Consequently $d\tilde{\phi} = 0$ since this is the only term of bi-degree $(2, n-1)$ multiplied by $0 \oplus \phi_0$; one then has $d\tilde{\phi} \oplus 0 = \text{int}(0 \oplus \phi_0) dm^*\phi$. By Lemma 6.1, $\tilde{\phi} \in E_0(\Delta^1_G)$. The contradiction completes the proof of Assertion (2). Assertion (3) is an immediate consequence of Assertion (2) since $E_0(\Delta^1_G) = \{0\}$ if G is simply connected. □

7. Finite Fourier series for general left invariant metrics

Let ds^2_G be a left invariant metric on a compact Lie group G and let $ds^2_{G\times G}$ be a left invariant metric on $G \times G$. We impose no relationship between the two metrics; in particular, we do not assume that the multiplication map m is a Riemannian submersion any more; it is an interesting question in its own right when this is possible and we shall investigate this question in more detail in a subsequent paper.

We begin the proof of Theorem 1.6 with a technical result:

Lemma 7.1. *Let G be a compact Lie group. Let H be left invariant subspace of $L^2(\Lambda^p(G))$. Then there is a bi-invariant subspace $H_1 \subset L^2(\Lambda^p(G))$ which contains H so that* $\dim(H_1) \le \binom{\dim\{G\}}{p} \dim(H)^2$.

Proof. The lemma is immediate if $\dim(H) = \infty$ so we may suppose H is finite dimensional. By decomposing $H = \oplus_i n_i V_{\rho_i}$ into irreducible representations, we may assume without loss of generality that $H = V_\rho$ where V_ρ is an irreducible left representation space for G in the proof of Lemma 7.1. We apply the Peter–Weyl theorem and use Eq. (3.a). Let

$$H^p_\rho = \bigoplus_{|I|=p} H_\rho \cdot \Phi^I_L = \binom{\dim\{G\}}{p} \dim(V_\rho) V_\rho.$$

Then $H \subset H^p_\rho$. Since left and right multiplication commute, $H^p_\rho \cdot R_g$ is isomorphic to H^p_ρ for any $g \in G$. Since H^p_ρ contains all representations isomorphic to V_ρ, $H^p_\rho \cdot R_g = H^p_\rho$ is right invariant as well. □

Let $\phi \in E_\lambda(\Delta^p_G)$ be an eigen-p-form. Apply Lemma 7.1 to choose a subspace $H_1 \subset L^2(\Lambda^p(G))$ which is left and right invariant under the action of G, which contains $E_\lambda(\Delta^p_G)$, and which satisfies:

$$\dim(H_1) \le \binom{\dim\{G\}}{p} \dim\{E_\lambda(\Delta^p_G)\}^2.$$

Let $\tilde{m}(a,b) = ab^{-1}$. Then $\tilde{m} L_{g_1,g_2} = L_{g_1} R_{g_2^{-1}} \tilde{m}$. Consequently $\tilde{m}^* H_1$ is a finite dimensional subspace of $L^2(\Lambda^p(G \times G))$ which invariant under left multiplication in the group. Apply Lemma 7.1 to choose a subspace $H_2 \subset L^2(G \times G)$ which is left and right under the action of $G \times G$, which contains $\tilde{m}^* H_1$, and which satisfies:

$$\dim(H_2) \le \binom{2\dim\{G\}}{p} \binom{\dim\{G\}}{p}^2 \dim\{E_\lambda(\Delta^p_G)\}^4.$$

Set $\psi(x,y) = (x, y^{-1})$. Then $\psi^* H_2$ is still bi-invariant and in particular is left invariant. Since $m^*\phi \in \psi^* H_2$, Lemma 3.1 can now be applied to show that

$$\mu(m^*\phi) \le \binom{2\dim\{G\}}{p}^2 \binom{\dim\{G\}}{p}^2 \dim\{E_\lambda(\Delta^p_G)\}^4.$$

Theorem 1.6 now follows. □

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References

[1] M. Boiteux, Theory of non-bijective canonical-transformations in mechanics: application to the Coulomb problem, J. Math. Phys. 23 (1982) 1311–1314.
[2] P. Gilkey, Asymptotic Formulae in Spectral Geometry, CRC Press, 2004.
[3] P. Gilkey, J. Leahy, J.H. Park, Spectral Geometry, Riemannian Submersions, and the Gromov–Lawson Conjecture, Chapman & Hall/CRC, 1999.
[4] P. Gilkey, J. Leahy, J.H. Park, The spectral geometry of the Hopf fibration, J. Phys. A 29 (1996) 5645–5656.
[5] B. Hall, Lie Groups, Lie Algebras and Representations, Springer-Verlag, New York, 2003.
[6] K. Hofmann, S. Morris, The Structure of Compact Groups, Walter de Gruyter, Berlin, 1998.
[7] M. Kibler, On the use of the group SO(4,2) in atomic and molecular physics, Mol. Phys. 102 (2004) 1221–1229.
[8] D. Lambert, M. Kibler, An algebraic and geometric approach to non-bijective quadratic transformations, J. Phys. A. Math. Gen. 21 (1988) 307–343.
[9] J. Lee, Riemannian Manifolds: An Introduction to Curvature, Springer-Verlag, 1997.
[10] M. Lachièze-Rey, S. Caillerie, Laplacian eigenmodes for spherical spaces, Classical Quantum Gravity 22 (2005) 695–708.
[11] E. Proctor, Isospectral metrics and potentials on classical compact simple Lie groups, Michigan Math. J. 53 (2005) 305–318.
[12] A. Ranjan, Riemannian submersions of compact simple Lie groups with connected totally geodesic fibres, Math. Z. 191 (1986) 239–246.
[13] D. Schueth, Isospectral manifolds with different local geometries, J. Reine Angew. Math. 534 (2001) 41–94.